Mental Calculation Strategies of a Student Attending a Special School for the Intellectually Disabled

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Pat was a 19-year-old attending a Special School for the Intellectually Disabled in Indonesia. She was interviewed by the first author regarding her mental calculation strategies when solving 1- and 2-digit addition and subtraction problems. Results indicate that she was able to see ten as a unit composed of ten ones and was facile in using standard written algorithms: addition with or without carrying and subtraction with or without borrowing. Her mental calculation strategies were influenced by the taught standard written algorithms. These algorithms seem to be counter-productive. However, with appropriate supports, she might have a potential to be an accurate and flexible mental calculator.

Following the results of research which suggests learning the standard written algorithm (Kamii,1998) can be harmful, and the awareness of the importance of mental calculation strategies in adult daily life (Northcote & McIntosh,1999), school curriculums in countries such as UK, US, and The Netherlands have emphasised the teaching of mental calculation strategies as an important part of teaching mathematics in schools. The new Australian national curriculum for mathematics also states that the use of mental-computation strategies should be developed in all stages. However, curriculums in other countries such as Indonesia (Badan Standard Nasional Pendidikan, 2006), do not mention mental calculation strategies. In our observations many teachers suggest that students should memorise the basic facts for solving 1-digit addition and subtraction problems, and use standard written algorithms for addition with or without carrying and subtraction with or without borrowing to solve 2-digit addition and subtraction problems at the second grade in regular schools.

A special school is a school catering students with special educational needs, for example, students with an intellectual or physical disability. Similar to students at regular schools, students at special schools in Indonesia also learn the basic mathematics needed to improve their ability to live independently. For example, the curriculum guidelines at special schools for the intellectually disabled advocate the teaching of standard written algorithms during the primary levels, and that the algorithms should be adjusted to take account of students' needs and abilities. Research on mental calculation strategies of those students is lacking even though there is a need to understand how these students calculate mentally. This paper describes the mental calculation strategies of a student attending a special school for the intellectually disabled and discusses how the taught standard written algorithm might affect her mental strategies. The paper concludes with recommendations for what teachers can do to help such students to enhance their mental calculation strategies.

Literature Review

Thompson (1999) explains that mental arithmetic is not the same as mental calculation. Mental arithmetic is connected with mental recall. It might be automatic and not involve new understanding about numbers. On the other hand, mental calculation or mental

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computation is defined by many mathematics education researchers and practitioners as calculating in the head with understanding, and without external devices. This paper defines mental calculation strategies in a similar way. In Indonesia, there is a broad term "mental aritmetika" which is a literal translation of "mental arithmetic". However, in Indonesia, the term "mental aritmetika" is used differently from the way "mental arithmetic" is used in Anglophone countries. It is used to describe calculation methods using a mental image of an abacus.

Upon listing the advantages and disadvantages of teaching standard written algorithms in the early years of primary school, Clarke (2005) argues that developing concepts and strategies for mental computation is far more important in the early years. Research suggests that students can invent mental calculation strategies (Carpenter et al., 1998), including those with an intellectual disability (Baroody, 1996). However, there is also a strong argument that, rather than teach mental calculation strategies, students should be left to invent those strategies and that teaching mental calculation strategies has a variety of outcomes for students (Murphy, 2004).

A wide range of mental calculation strategies has been identified in the literature. These strategies can be divided into two categories: mental calculation strategies for 1-digit number problems such as counting and using derived facts; and strategies for multi-digit number problems such as jumping, splitting and compensation strategies. Table 1 shows examples of mental calculation strategies found in the literature.

Table 1.
Examples of Mental Calculation Strategies for 1- and 2-Digit Number Problems

For	Labels		Example for addition or subtraction
1-digit	Counting	Counting on	5 + 3→ 6, 7, 8 answer is 8
		Counting back	8 – 5→7, 6, 5, 4, 3 answer is 3
	Deriving facts	Doubles	5 + 6→5 + 5+1
		Bridging through ten	7 + 8→7 + 3 = 10, 10 + 5 = 15 13 – 5→13 – 3 = 10, 10 – 2 = 8
		Compensation	9 + 5→because 10 + 5 = 15, so 9 + 5 = 14
2-digit		1010 (split)	24 + 13→20 + 10 = 30, 4 + 3 = 7→30 + 7 = 37
			45 – 21→40 – 20 = 20, 5 – 1 = 4→20 + 4 = 24
		N10 (Jump)	36 + 25→36 + 20 = 56→56 + 5 = 61
			52 – 17→52 – 10 = 42→42 – 7 = 35
		Combination	36 + 25→30 + 20 = 50→50 + 6 = 56→56 + 5 = 61
			52 – 17→50 – 10 = 40→40 + 2 = 42→42 – 7 = 35
		Compensation	52 – 19→(52)50 – 19 = 31→31 + 2 = 33
	Mental image of a standard written algorithm		
	Mental image of an abacus		

Beishuizen, 1993; Thompson, 1999, Clark, 2008

Heirdsfield and Cooper (2002) compared the mental calculation strategies used by two students. They found that even though both students were accurate, one used flexible strategies and the other inflexible strategies. The inflexible strategies were based on a mental image of a standard written algorithm while the flexible strategies involved a variety of methods. Furthermore, Heirdsfield and Cooper (2004) argued that cognitive, metacognitive and affective factors operate differently for students with flexible strategies, compared with those with inflexible strategies.

Method

Participant and Study Site

At the time of interview, Pat was in Year 11, that is, one year before school completion. The first author met her during a visit to a special school in Yogyakarta district, Indonesia. The purpose of the visit was to understand how students at special schools learn mathematics. A teacher introduced the first author to Pat. Pat had been in the special school since her first grade and she enjoys learning. Even though labelled as intellectually disabled, Pat was very fluent in communication and was not shy with a new person. She agreed to be interviewed regarding her number knowledge and her strategies in solving calculation problems. The first author also explained to her that the interview would be videotaped and the results might be published. However, in order to protect her privacy, the publication will not use her real name.

The Interview

The main purpose of the interview was to understand Pat's strategies for calculating mentally. The interview took about 25 minutes divided into three parts. In the first part, Pam's knowledge of numerals: ability to read and write numerals was assessed. In part two, Pam was asked to solve 1-digit addition and subtraction problems and in part 3, Pam was asked to solve 2-digit addition and subtraction problems. Problems were written on a piece paper and presented to her one by one and she was asked to solve the problems mentally. The interviewer asked her to describe her strategies and sometimes asked her to think aloud. The interview process was conducted as flexibly as possible. It was more an informal chat in a library during break time than a formal test. During the interview, her comfort and ease was taken seriously, in order to make sure that she could perform at her highest levels in solving calculation problems. The interviewer also assessed her number knowledge and her base-ten arithmetical strategies using tasks adapted from Wright, Martland, and Stafford (2006, pp 166-7).

Data Analysis

The main sources of data were the videotaped record of the interview and Pat's written work. Videotaping allowed for retrospective analysis. The recording was watched several times in order to ensure that the mental calculation strategies used by Pam were determined accurately. Table 1 enabled the determination of her strategies. However, we also choose an open approach in case Pat used strategies not listed in Table 1.

Results and Discussion

Pat's Knowledge of Numerals

Prior to investigating Pat's mental calculation strategies, her ability to identify and write numerals was assessed. Numerals in the range 0 to 9 were presented to her in the following pseudorandom order —3, 2, 0, 8, 7, 5, 9, 1, 4, and 6, and she correctly identified all 10 numerals, including correctly saying 'zero' for '0'. For 2-digit numerals in the following pseudorandom order —11, 13, 59, 25, 98, 50, 23, 48, 83, 20, 19, 77, 12, 15, and 21, Pat identified all correctly, and always stated the complete number words such as saying *lima puluh Sembilan* (fifty-nine) for 59. For the following 3-digit numerals —604, 800, 101, 710, 234, 543, 121, 110, 456, 666, and 4-digit numerals—1112, 5000, 8888, 4560, 2008, 3500, 1000, 7245, 5080, 9074, Pat also identified all correctly and completely, such *tujuh ribu dua ratus empat puluh lima* (seven thousand, two hundred, forty-five) for 7245. Pat's ability in writing 5- and 7-digit numbers was also assessed. The interviewer said each number in turn—70536, 156230 and 1200000 and she wrote them correctly. She was correct in all those the tasks. Figure 1 shows the numbers she wrote.

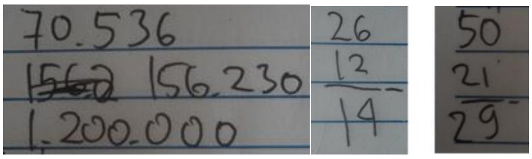

Figure 1. Pat's written work on writing numbers and on subtraction 26 – 12 and 50 – 21

Pat has good understanding of numbers and numerals up to 7-digits. She also always says multi-digit numbers words completely. In my experience, some students says 59 as *lima Sembilan* (six nine), not *lima puluh Sembilan* (sixty nine). Thus some students omit the ten (*puluh*) which is crucial indicator of place value. For multi-digit numbers such 7245 some students also say *tujuh dua empat lima* (seven two four five) *or tujuh ribu dua ratus empat lima (seven thousands and two hundred four five)* instead of the correct and complete number words *tujuh ribu dua ratus empat puluh lima* (seven thousands and two hundreds forty five). This indicates that Pat has an awareness of place value: tens, hundreds, thousands, etc. As well, she is facile in writing numerals with up to seven-digits. Her strong number knowledge should constitute an adequate basis for solving calculation problems. Even though number sense may not sufficient for accurate mental calculation (Heirdsfield and Cooper, 2004), we believe that Pat's strong number knowledge may provide a solid and useful basis enabling her to be a proficient and flexible mental computer.

Pat Solved 1-Digit Addition and Subtraction Problems

Following the assessment of identifying and writing numerals, Pat was asked to solve 1-digit number problems. The interviewer presented a contextual problem: If you have 5 counters and you get 6 more, how many do you have altogether? Pat answered "eleven" very quickly. However when asked how she worked it out, she could not give any explanation. Also, when the interviewer asked a similar question: If you have seven counters and you get eight more, how many are there altogether; she again could not explain her answer, even though she answered the question correctly and quickly.

At this point the interviewer changed the way she presented the tasks. She presented Pat with four horizontal number problems, each written on a separate card—7 + 8, 5 + 6, 8 + 7, and 9 + 4. Pat answered quickly and correctly for all four tasks. However, she was not able to explain how she obtained her answers, merely saying she already knew that.

After attempting unsuccessfully to understand Pat's strategies for solving 1-digit addition problems, the interviewer tried to understand Pat's strategies for solving subtraction problems. Pat was asked to solve 13 – 5 which was also written on a card. She correctly answered eight. Following is the transcription when the interviewer tried to probe Pat's strategy.

Interviewer : *Shows 13 − 5 written on a piece of paper.*
Pat : 8
Interviewer : How do you know the answer is 8?
Pat : Silent for about 30 seconds.
Interviewer : Now, imagine you are a teacher and you want to explain how to solve 13 − 5 to a student? How would you explain that?
Pat : 13 − 5…13 (pause) 13…12…10... (pause), 13 – 3 = 10 and 10 – 2 = 8…so the answer is 8.
Interviewer : Okay.

The above transcription shows that when asked about her strategy in solving the 1-digit subtraction problem 13 – 5, it seems that she started using a counting back strategy, and then she ceased counting. She then seemed to use a bridging-through-ten strategy. It seems that Pat has knowledge of structuring numbers around ten, and also she is facile in the partition of numbers. She knows that 5 can be partitioned into 3 and 2 and applies that knowledge to solve 13 – 5.

The interviewer then presented the addition problems again and this time Pat explained her strategies. Table 2 shows the tasks and the strategies she described.

Table 2
Pat's Strategies in Solving 1-Digit Addition Problems

Problems	Answer	How did you know the answer?	Do you have another way?
9 + 7	16	Because 9 + 1 = 10 and 10 + 6 = 17	No
9 + 4	13	Because 9 + 1 = 10 and 10 + 3 = 13	No
5 + 6	11	Because 5 + 5 = 10 and 10 + 1 = 11	No
7 + 8	15	Because 7 + 3 = 13 and 10 + 5 = 15	No

Table 2 shows that when the interviewer went back to probe her strategies for solving 1-digit addition problems, she also explained these in terms of a bridging through ten strategy. While there is no evidence that Pat has been taught this strategy previously, it seems that Indonesian number words which are structured around ten have been a support for Pat's success. Indonesian number words are similar to Bruneian and Malaysian number words, which are structured around ten (Nwabueze, 2001). A study by Fuson and Kwon (1992) also shows that number structure around ten supports Korean children in learning efficient decomposition strategies for solving 1-digit number problems. This case supports the suggestion of Mulligan, Vale and Stephens (2009) that understanding and developing structure is an important part for mathematics learning.

Pat Solved 2-Digit Addition and Subtraction Problems

Prior to investigating Pat's strategies in solving 2-digit addition and subtraction mentally, Pat's ability with base-ten arithmetical strategies was assessed. The interviewer put down a ten-strip and asked how many dots there are. Pat did not count by ones. Rather, she put her forefinger in the middle of the strip and after being silent for 3 seconds, she said "ten". Following this the interviewer put down one ten-strip and asked, "Now, how many dots are there?" Pat did not count again. Rather, she directly answered "20". When the interviewer placed more ten-strips one at a time, she was able to answer correctly without counting by ones: "30, 40, 50, 60, … 100".

Following this, the interviewer presented the "uncovering task" in which 1- and 2-digit numbers in the form of dot-strips are progressively uncovered. Upon each uncovering, Pat answered how many dots in all. The following is the transcription.

Interviewer : *If there is 10 and I add 3?*
Pat : *13!* (quickly)
Interviewer : *If there is 13 and I add 20, how many dots are there altogether?*
Pat : *33!*
Interviewer : *How do you know that it is 33?*
Pat : *Because there are 10, and 10 and 10 and 3.*
Interviewer : *How about if we add 4 more?*
Pat : *37!* (quickly)
Interviewer : *and 3 more*?
Pat : *40!* (quickly)
Interviewer : *and 10 more?*
Pat : *50!* (quickly)
Interviewer : *and 12 more?*
Pat : (silent a moment) *62!*
Interviewer : *Why?*
Pat : *Because 50 plus 10 is 60 and there are 2 more. So 62.*

At this point, Pat's strategies in solving 2-digit addition problems were assessed. There were four addition problem presented: 16 + 10, 20 + 21, 38 + 24 and 29 + 18. Table 3 show Pat's strategies in solving these problems.

Table 3
Pat's Strategies in Solving 2-Digit Addition Problems

Problems	Answer	How did you know the answer?	Do you have another way?
16 + 10	26	Because 16 + 4 = 20 and 20 + 6 = 26	Yes, 1+1=2 and there is 6
20 + 21	41	Because 2 + 2 = 4 and 0 + 1 = 1	No
38 + 24	62	8 + 4 = 12, 3 + 2 = 5 … (pause) … 62	No
29 + 18	47	9 + 8 = 17…1 + 2 = 3…3 + 1 = 4 47	No

Results from the base-ten arithmetical strategies assessment and the "uncovering task" indicate that Pat was able to see ten as a unit composed of ten ones. However, when she was asked to solve horizontal, written number tasks, her strategies changed. Initially, Pat explained that 16 + 10 = 26 because 16 + 4 = 20, suggesting that even on this task she used knowledge that number is structured around tens. When asked 'Do you have another way', she said that 16 + 10 will make 26 because 10 plus 10 is 20 and there are 6 more. Thus she seemed to use the split strategy. However, for the next tasks she apparently used a mental image of standard written algorithm. Her written task, as shown in Figure 1, indicates that she is a facile user of standard, columnar, written algorithm.

Pat's strategies in solving 2-digit subtraction problems were now assessed. Table 4 shows her strategies in solving 16 − 10, 26 − 12, 50 − 21, 30 − 19, 31 − 23, and 41 − 24. Finally, Pat was asked to solve two 2-digit subtraction problems and she was permitted to use pencil and paper. Figure 1 also shows Pat's written work on subtraction. When she was asked whether she had other strategies for solving subtraction number problems, she said "No'.

For 2-digit subtraction problems, Pam also used mental image of standard written algorithm, she always answered that she does not know another strategy. While her results indicate that her ability to increment and decrement numbers by tens and ones might play an important role in her mental calculation strategies, this was not the case for Pat. She seems to abandon her thinking. She was not aware that 31 – 23 = 18 is incorrect. For this task she used a 'buggy algorithm' or procedure which is partially correct, and chose not to continue the interview. This is in line with the findings of Hatano, Amaiwa, and Inagaki (1996) that 'buggy algorithms' can be an attractive variant for students and they may rely more on buggy algorithms when asked to solve many problems, not because they do not know the correct procedure, but because it seems more efficient.

Table 4
Pat's Strategies in Solving 2-Digit Subtraction Problems

Problems	Answer	How did you know the answer?	Do you have another way?
16 – 10	6	Because 6 – 0 = 6	No
26 – 12	12	Because 6 – 2 = 4 and 2 – 1 = 1	No
50 – 21	29	We can't subtract 1 from 0, so we have to borrow from 5, so 10 1 = 9, and 4 – 2 = 2, so 29	No
30 – 19	11	We can't subtract 9 from 0 so we borrow 1 from 3, and then 10 - 1 = 9. There is still 2 left and subtract with 1, so 11.	No
31 – 23	18 (incorrect)	We can't subtract 3 from 1, so we borrow 1 from 3, so 11 – 3 = 8. And then 3 – 2 = 1, so 18.	No
41 – 24	-	Refused to answer	

Pat appears to possess both procedural and conceptual knowledge needed to be a proficient and accurate mental calculator. As apparent with her base-ten arithmetical strategy assessment results she seems just on the edge of developing flexibility and efficiency. She is likely to develop good number sense, and mental computation can facilitate number her sense when she is encouraged to be flexible, even though flexibility and number sense are neither necessary nor sufficient for accuracy in mental computation (Heirdsfield & Cooper, 2004). A study by Callingham (2005) suggests that students who were procedural in their approach gain significant success after a strategy-based intervention. Pat might be a good candidate for such strategy-based intervention. Furthermore, her teacher should play an important role in developing her confidence using the knowledge she already possesses. Peters, Smedt, Torbeyns, Verschaffel, and Ghesquière (2014) suggest typical special education practice, including practice for students with an intellectual disability, to not only focus on routine strategies, but also to provide opportunities for students with an intellectual disability to develop their own strategies. Using a number line also may enable her to reduce her reliance on strategies related to standard written algorithms (Bobis, 2007) and help her to be a facile and flexible mental calculator.

Conclusion

This case study presents an example of a student who is registered as intellectually disabled and has mastered the standard written algorithm but these taught algorithms seem to inhibit her ability to make sense of her solutions especially when she was asked to solve 2-digit subtraction problems. Since this is just a one case the results should not be generalised. This is a limitation of the study. However, this case might be useful in supporting Clarke's (2005) claim that there is a possible detrimental effect of teaching written algorithms in the early years, on children's mental strategies and number sense.

While Pat appears to be able to use bridging through ten strategies to solve 1-digit addition and subtraction problems, and a splitting strategy to solve 2-digit addition problems, her strategy choice for later tasks was a familiar mental image of standard algorithm. Her advanced strategies for 1-digit addition and subtraction, her understanding of base ten or place value, and her strong number knowledge which may provide a sufficient basis for her to develop flexible mental calculation, seems not been utilised.

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